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INCIDENCE THEOREMS ON MANIFOLDS

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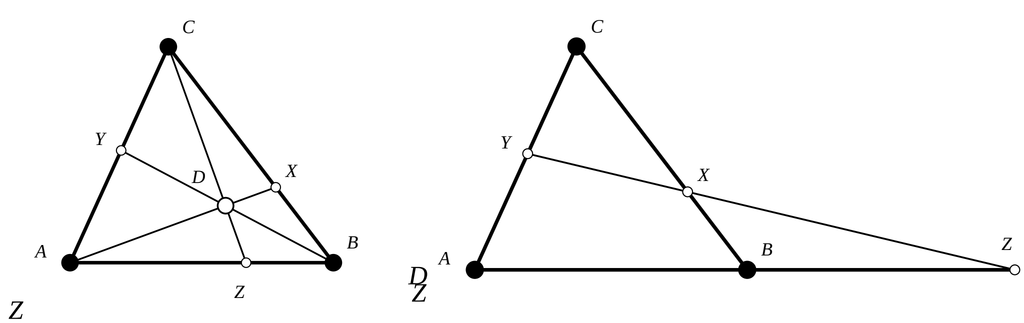
1. Introduction

This presentation focuses on structural aspects of algebraic proofs for incidence theorems. Several approaches to proving incidence theorems by algebraic methods are compared and it is shown that all of them are essentially equivalent. A special role is played by proofs that are generated by joining together many distinct copies of Ceva's or Menelaus' Theorem. These proofs can be equipped with an underlying manifold structure. The fact that one "indeed has a proof" corresponds to the fact that the manifold has no boundary. Thus additional structural insight is obtained by directly associating cyclic cancellation patterns in proofs to cyclic structures on manifolds. The correspondence between manifolds and algebraic proofs makes use of essential facts that occur in the theory of *Tutte Groups for matroids* that was introduced by Dress and Wenzel in 1984 (see for instance [2]). As applications of these new structural insights one can for instance generate a complete classification of liftable rhombic tilings with three directions.

2. Ceva's and Menelaus' Theorem

Here we want to sketch right away the relation between incidence theorems and manifolds. At first sight the presented approach to incidence theorems seems to be very special but indeed our main theorem will show that this approach is always applicable when certain other classical proving methods (Area Method á la Shephard & Grünbaum [3], Biquadratic Final Polynomials [1]) apply.

Our main protagonists are the theorems of Ceva and of Menelaus. Ceva's Theorem states that if in a triangle the sides are cut by three concurrent lines that path through the corresponding opposite vertex the product of the three (oriented) length rations along each side equals 1. Menelaus' Theorem states the this product is -1 if the cuts along the sides come from a single line.



Ceva's Theorem: $\frac{|AZ|}{|ZB|} \cdot \frac{|BX|}{|XC|} \cdot \frac{|CY|}{|YA|} = 1$ Menelaus' Theorem: $\frac{|AZ|}{|ZB|} \cdot \frac{|BX|}{|XC|} \cdot \frac{|CY|}{|YA|} = -1$

There is an immediate proof of these facts if one considers the ratios along each side as a quotient of triangle areas. Let $[ABC]$ denote the (oriented) area of the triangle ABC. We obviously have

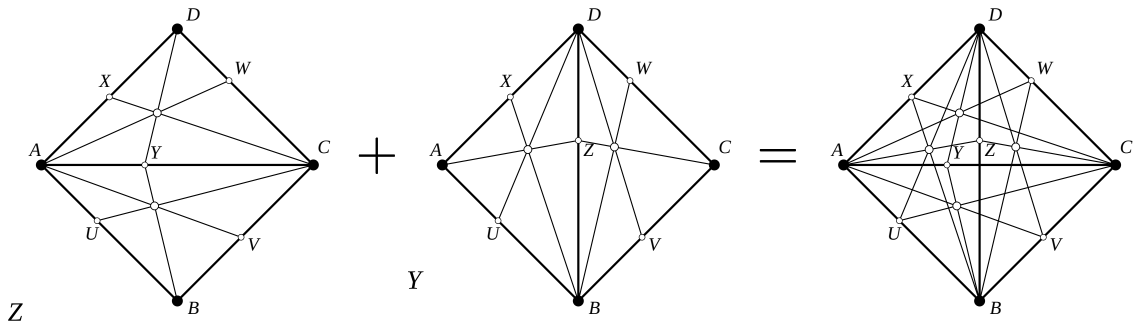
$$\frac{[CDA]}{[CDB]} \cdot \frac{[ADB]}{[ADC]} \cdot \frac{[BDC]}{[BDA]} = -1.$$

Observing $\frac{[CDA]}{[CDB]} = -\frac{|AZ|}{|ZB|}$ (and the corresponding equations for the other sides) we get Ceva's Theorem. Similarly a proof of Menelaus' Theorem is derived. Consider the special line as being generated by two points D and E. We have

$$\frac{[DEA]}{[DEB]} \cdot \frac{[DEB]}{[DEC]} \cdot \frac{[DEC]}{[DEA]} = 1,$$

This is in essence Menelaus' Theorem.

2.1. **Gluing together Ceva and Menelaus Configurations.** Now take any triangulated oriented 2-Manifold. This manifold serves as a kind of *frame* for the construction of an incidence theorem. Consider this manifold as being realized by flat triangles (it does not matter if these triangles intersect, coincide or are coplanar as long as they represent the combinatorial structure of the manifold). Let us be concrete and take the projection of a tetrahedron $(ABCD)$ to $\mathbb{R}^2$. Now choose Points U,V,W,X,Y,Z one on each of the edges of the tetrahedron. Assume that for three of the faces these points satisfy Ceva's condition. Then they automatically satisfy it also for the last face — an incidence theorem.



The proof of this theorem is almost obvious from the algebraic characterization of Ceva's condition. Consider the following formula

$$\left(\frac{|AU|}{|UB|}\cdot\frac{|BV|}{|VC|}\cdot\frac{|CY|}{|YA|}\right)\cdot\left(\frac{|CW|}{|WD|}\cdot\frac{|DX|}{|XA|}\cdot\frac{|AY|}{|YC|}\right)\cdot\left(\frac{|AX|}{|XD|}\cdot\frac{|DZ|}{|ZB|}\cdot\frac{|BU|}{|UA|}\right)\cdot\left(\frac{|BZ|}{|ZD|}\cdot\frac{|DW|}{|WC|}\cdot\frac{|CV|}{|VB|}\right)=1.$$

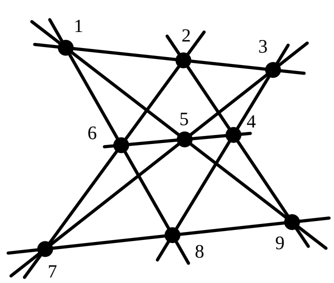
This formula is obviously true since exactly all terms of the numerator occur in the denominator as well. On the other hand each of the Factors being 1 states the Ceva condition for one of the faces. Thus three of these conditions imply the last one. The essential fact about the structure of this proof is that whenever two faces meet in an edge the two corresponding ratios cancel. In general we obtain:

> For any triangulated 2-manifold choose a point on each edge such that for every face either a Ceva or a Menelaus condition is generated. If altogether an even number of Menelaus configuration is involved, then the last condition is met automatically.

3. Comparison with other methods

This method of generating an incidence theorem may seem rather special. Nevertheless it is a mater of fact that many classical incidence theorems can be generated this way. It is possible to transfer other proving techniques into this structure. In particular proofs generated by the "Area Method" (in the sense of Grünbaum & Shephard) or by the method of "biquadratic final polynomials" can be translated into Ceva/Menelaus proofs. The proof of this fact makes use of concepts developed in the context of Tutte groups.

The method of biquadratic final polynomials considers for each collinearity (A,B,C) all equations of the form $[ABX][ACY]=[ABY][ACX]$. Searching for the suitable dependencies among these equations is a method of automatically finding algebraic proofs. Again we will exemplify the concept by a specific example. Consider the well known Pappos theorem. A biquadratic proof is given in the picture.



COLL(123)	$\Longleftrightarrow$	$[124][137]=[127][134]$
COLL(159)	$\Longleftrightarrow$	$[154][197]=[157][194]$
COLL(168)	$\Longleftrightarrow$	$[184][167]=[187][164]$
COLL(249)	$\Longleftrightarrow$	$[427][491]=[421][497]$
COLL(456)	$\Longleftrightarrow$	$[457][461]=[451][467]$
COLL(348)	$\Longleftrightarrow$	$[487][431]=[481][437]$
COLL(267)	$\Longleftrightarrow$	$[721][764]=[724][761]$
COLL(357)	$\Longleftrightarrow$	$[751][734]=[724][731]$
COLL(789)	$\Longleftrightarrow$	$[781][794]=[724][791]$

Under the non-degeneracy assumption that $1,4,7$ are not collinear the first eight equation correspond to the hypothesis of the theorem. Multiplying everything on the left and everything on the right of these equations and canceling terms that occur on both sides produces exactly the last equation, which is the conclusion.

Usually this kind of proof does not carry a natural manifold structure (as the Ceva/Menelaus proofs do). The following argument exemplifies that nevertheless each such proof can be transferred into a Ceva/Menelaus proof. We can write any biquadratic proof in multiplicative form. For instance the Pappos proof becomes:

$$\frac{[124][137]}{[127][134]}\cdot\frac{[154][197]}{[157][194]}\cdot\frac{[184][167]}{[187][164]}\cdot\frac{[427][491]}{[421][497]}\cdot\frac{[457][461]}{[451][467]}\cdot\frac{[487][431]}{[481][437]}\cdot\frac{[721][764]}{[724][761]}\cdot\frac{[751][734]}{[754][731]}\cdot\frac{[781][794]}{[784][791]}=1$$

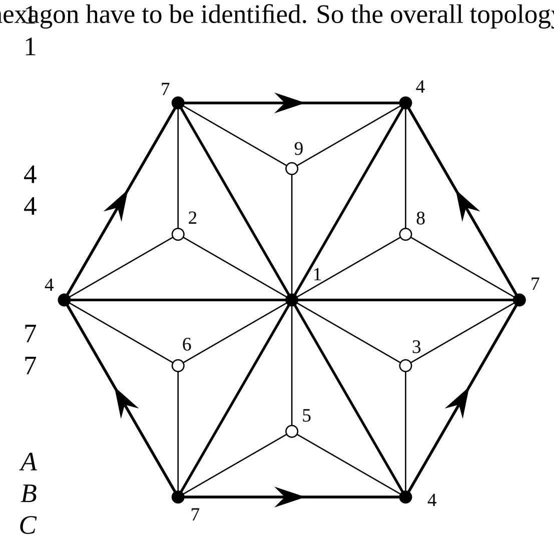
The nine blocks correspond to the nine biquadratic equations That fact that we have a proof implies that each bracket in the numerator occurs in the denominator as well. Now a *Homotopy theorem on Mauer graphs* shows that each bracket quotient with this property (where a ratio of two brackets is considered as an indecomposable symbol in an abstract group) can be generated by multiplication of finitely many Ceva/Menelaus conditions. In general the expression may get (considerably) longer however in the case of Pappos theorem this can be done by a simple rearrangement of the ratios in the expression. So we first decompose each binomial ratio in two factors:

$$\frac{[124]}{[127]}\cdot\frac{[137]}{[134]}\cdot\frac{[154]}{[157]}\cdot\frac{[197]}{[194]}\cdot\frac{[184]}{[187]}\cdot\frac{[167]}{[164]}\cdot\frac{[427]}{[421]}\cdot\frac{[491]}{[497]}\cdot\frac{[457]}{[451]}\cdot\frac{[461]}{[467]}\cdot\frac{[487]}{[481]}\cdot\frac{[431]}{[437]}\cdot\frac{[721]}{[724]}\cdot\frac{[764]}{[761]}\cdot\frac{[751]}{[754]}\cdot\frac{[734]}{[731]}\cdot\frac{[781]}{[784]}\cdot\frac{[794]}{[791]}=1$$

After this we rearrange the ratios to get a Ceva/Menelaus proof. In this expression each (big) factor corresponds to a Ceva configuration.

$$\frac{[124][721][427]}{[127][724][421]}\cdot\frac{[137][431][734]}{[134][437][731]}\cdot\frac{[154][751][457]}{[157][754][451]}\cdot\frac{[197][491][794]}{[194][497][791]}\cdot\frac{[184][781][487]}{[187][784][481]}\cdot\frac{[167][461][764]}{[164][467][761]}=1$$

The fact that no additional Ceva configurations are necessary corresponds to the fact that already the original biquadratic proof carried a manifold structure. The following diagram illustrates the manifold structure of the proofs. The six big equilateral triangles in the picture corresponds to the six Ceva configurations that arise in the proof. Opposite sides of the hexagon have to be identified. So the overall topology of the proof is a torus.



In each triangle the point in the middle corresponds to the Ceva point. The interior structure of the triangles forms nine rhombi (after the identification). Each rombus corresponds to one of the original equations of the biquadratic proof.

4. Further investigations and applications

Furthermore it will be demonstrated, how the structural insight can be used to completely classify incidence theorems that involve only three bundles of parallel lines. This leads to a complete classification of rhombic tilings with three directions with respect to liftability. As a further enhancement it is also demonstrated how the set of geometric primitives can be enriched to cover also statements about conics.

References

[1] J. Bokowski & J. Richter, *On the finding of final polynomials, Eur. J. Comb.*, **11** (1990), 21–34.
[2] A. Dress & W. Wenzel, *On combinatorial and projective geometry, Geom. Dedicata*, **34** (1990), 161–197.
[3] G.C. Shephard & B. Grünbaum, *Ceva, Menelaus, and the area principle, Mathematics Magazine*, **68** (1995), 254–258.